Total Colorings of Planar Graphs with Large Maximum Degree

O. V. Borodin,[1,*] A. V. Kostochka,[2,†] and D. R. Woodall[3]
[1] *NOVOSIBIRSK STATE UNIVERSITY*
NOVOSIBIRSK, 630090, RUSSIA
[2] *INSTITUTE OF MATHEMATICS*
SIBERIAN BRANCH, RUSSIAN ACADEMY OF SCIENCES
NOVOSIBIRSK, 630090, RUSSIA
[3] *DEPARTMENT OF MATHEMATICS*
UNIVERSITY OF NOTTINGHAM
NOTTINGHAM, NG7 2RD, ENGLAND

Received April 29, 1996

Abstract: It is proved that a planar graph with maximum degree $\Delta \geq 11$ has total (vertex-edge) chromatic number $\Delta + 1$. © 1997 John Wiley & Sons, Inc. J Graph Theory **26:** 53–59, 1997

Keywords: *total coloring, planar graphs, discharging*

1. INTRODUCTION

The *total chromatic number* $\chi'' = \chi''(G)$ of a graph G is the smallest number of colors that suffice to color the vertices and edges of G so that no two adjacent or incident elements have the same color. It is clear that $\chi'' \geq \Delta + 1$, and Behzad [1] and Vizing [12] conjectured that

* This work was carried out while the first author was visiting Nottingham, funded by Visiting Fellowship Research Grant GR/K00561 from the Engineering and Physical Sciences Research Council.
† The work of the second author was partly supported by grant 96-01-01614 of the Russian Foundation for Fundamental Research and by the Program DIMANET/PECO of the European Union.

 CCC 0364-9024/97/010053-07

$\chi'' \leq \Delta + 2$, for every graph G with maximum degree Δ. This conjecture was verified by Rosenfeld [10] and Vijayaditya [11] for $\Delta = 3$ and by Kostochka [7, 8, 9] for $\Delta \leq 5$. For planar graphs the conjecture was verified by Borodin [2] for $\Delta \geq 9$, and now remains open only for $6 \leq \Delta \leq 7$ [6]. But for planar graphs it is often possible to obtain the exact result $\chi'' = \Delta + 1$. We proved this in [3] for planar graphs with various combinations of maximum degree and girth. With no condition on the girth, Borodin [2] proved it for planar graphs with maximum degree $\Delta \geq 14$. In [4] we improved this to $\Delta \geq 12$ (and, moreover, with the list total chromatic number χ''_{list} in place of χ''). This paper is devoted to a proof of the following further improvement for χ''.

Theorem. If G is a planar graph with maximum degree $\Delta \geq 11$, then $\chi''(G) = \Delta + 1$.

In fact, we prove that if $\Delta \geq 11$ then $\chi''(G) \leq \Delta + 1$ for all planar graphs with maximum degree *at most* Δ, and we let $G = (V, E)$ be a counterexample to this more general result with $|V| + |E|$ as small as possible, so that every proper subgraph of G is totally $(\Delta + 1)$-colorable. It is easy to see that G is 2-connected, and hence has no vertices of degree 1.

2. REDUCIBLE CONFIGURATIONS

In this section we start the proof of the theorem by obtaining structural information about our minimal counterexample G, which shows that certain configurations are *reducible,* that is, they cannot occur in G. The degree of vertex v is denoted by $d(v)$, and a *d-vertex* is a vertex v with $d(v) = d$.

Lemma 1. (a) If uw is an edge of G with $d(u) \leq 5$, then $d(u) + d(w) \geq \Delta + 2 \geq 13$ (and so 2-vertices of G are adjacent only to Δ-vertices).

(b) The graph of all edges joining 2-vertices to Δ-vertices is a forest.

Proof. If G contains an edge uw with $d(u) \leq 5$ and $d(u) + d(v) \leq \Delta + 1$, then we can totally color $G - uw$ with $\Delta + 1$ colors (by the minimality of G), erase the color on u, and then color uw and u in turn, since the number of colors that we may not use is at most $(\Delta - 1) + 1 = \Delta$ for uw and $5 + 5 = 10 < \Delta$ for u. This contradicts the choice of G as a counterexample and so proves (a).

If (b) is false then G contains a cycle $v_1 v_2 \cdots v_{2k} v_1$ of even length such that $d(v_1) = d(v_3) = \cdots = d(v_{2k-1}) = 2$. Then, by the minimality of G, we can totally color $G - \{v_1, v_3, \ldots, v_{2k-1}\}$ with $\Delta + 1$ colors. Each edge of C now has at most $(\Delta - 2) + 1 = \Delta - 1$ colors that may not be used on it, hence at least two that may, and so the problem of coloring the edges of C is equivalent to coloring the vertices of an even cycle, given a choice of two colors at each vertex; it is well known [5, 13] that this is possible. The 2-vertices of C are now easily colored, and this contradicts the choice of G as a counterexample to the theorem, thereby proving (b). ∎

In Figure 1, vertices marked • have no edges of G incident with them other than those shown, and so have degree 2 or 3 in G.

Lemma 2. G has no subgraph isomorphic to either of the configurations in Figure 1 (a) and (b).

Proof. Suppose that G does contain one of these configurations. Choose a total coloring of $G - uv$ with $\Delta + 1$ colors, which exists by the minimality of G, and erase the colors on the vertices marked •. If we can prove that the resulting partial total coloring c can be extended to edge uv, then we will have a contradiction to the choice of G as a counterexample to the theorem,

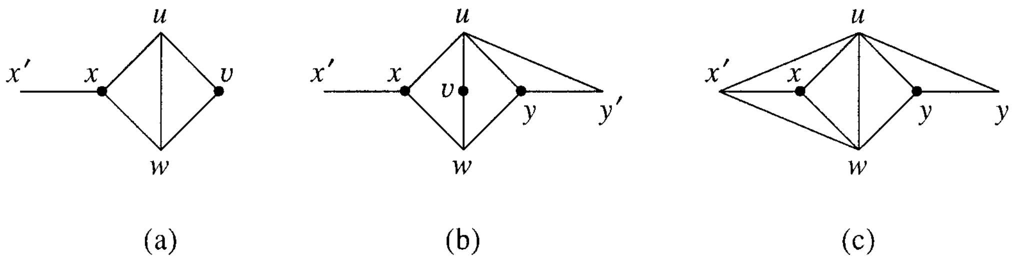

FIGURE 1. Some reducible configurations.

since vertices with degree at most 3 are easily colored. Let $c(vw) = 1$. We may suppose that color 1 is not present at u (that is, $1 \neq c(u)$ or $c(e)$ for any edge e incident with u), since if it is then some other color is not present at u and can be given to uv. The proof now divides.

(a) Since $1 = c(vw)$ is not present at u, the three edges in the path $vwux$ have three different colors, say 1, 2, 3 (in that order). Recolor these edges 1, 2, 1 or 2, 1, 2 according as $c(xx') = 2$ or $\neq 2$, and color uv with 3.

(b) We may suppose that $c(xx') = c(yy') = 1$, since otherwise we can give color $c(ux)$ or $c(uy)$ to uv and recolor ux or uy with 1. Thus the edges of path $vwyy'u$ have colors (w.l.o.g.) 1, 2, 1, 3 or 1, 2, 1, 2. In the former case, recolor these edges 1, 2, 3, 1 and color uv with 3. In the latter case, recolor the edges of path $vwyy'ux$ with 2, 1, 2, 1, 2 and give color $c(ux)$ to uv. ∎

Lemma 3. G has no subgraph isomorphic to the configuration in Figure 1(c).

Proof. Suppose it has. Let C denote the 4-cycle $uxwyu$. Color $G - E(C)$ and erase the colors on x, y. Choose this coloring c if possible so that $c(xx') \neq c(yy')$ and $c(uw) \in \{c(xx'), c(yy')\}$. We now show that this is possible.

Suppose first that $c(xx') = c(yy'), = 1$, say. If color 1 is present at both u and w then there are at least two colors available for each edge of C, which leads to a contradiction since an even cycle is list-edge-2-colorable (and x and y are then easily colored). Thus color 1 is absent at u or w, and we can ensure $c(xx') \neq c(yy')$ by swapping the color of xx' with that of $x'u$ or $x'w$ respectively. So from now on, let $c(xx') = 1$, $c(yy') = 2$.

Now suppose that $c(uw) \notin \{1, 2\}$. If we cannot change $c(uw)$ to 1 or 2 then each of these colors is present at at least one of u, w. If both are present at u (say), then two other colors are absent at u and we can color wx, wy first and then ux, uy. If however 1 (only) is present at u and 2 (only) at w, then after coloring ux, wy with 2, 1, there remains a color for each of uy, wx. This contradiction shows that we may suppose $c(uw) \in \{1, 2\}$.

Let $O(v)$ denote the set of colors absent at vertex v. If $c(uw) = 1$, it is now easy to complete the coloring unless $|O(u)| = |O(w)| = 2$ and $2 \in O(u) \cap O(w)$, in which case we might have to give color 2 to both ux and wx. The same applies with 1 and 2 interchanged. In either case, $|O(u) \setminus \{1, 2\}| = |O(w) \setminus \{1, 2\}| = 1$, say $O(v) \setminus \{1, 2\} = \{\bar{c}(v)\}$ $(v = u, w)$. By symmetry (since we shall make no further use of $x'w$) we may suppose that $c(uy') = 3 \neq \bar{c}(w)$. Now we can color or recolor the edges of the trail $wyy'uxwuy$ with colors 2, 3, 2, 3, $\bar{c}(w)$, 1, $\bar{c}(u)$. Finally, x and y are easily colored. ∎

Lemma 4. No face of G has more than one 2-vertex in its boundary.

Proof. Suppose that v, x are 2-vertices in the boundary of a face f. If v, x are separated by at least two other vertices each way round the boundary, then we can simply identify them; the resulting simple planar graph is totally $(\Delta + 1)$-colorable by the minimality of G, and this gives a total-$(\Delta+1)$-coloring of G. So suppose that u, v, w, x, y are consecutive vertices in the boundary of f. By Lemma 1 (a) and (b), $u \neq y$. So identifying v with x and deleting one of the edges vw and wx, say vw, gives a simple planar graph, which has a total-$(\Delta + 1)$-coloring c by the minimality of G. Apply this coloring to G. If $c(xy)$ is absent at w, give this color to edge vw; if not, give edge vw color $c(wx)$ and recolor wx with a color that is absent at w (and so different from $c(xy)$), finally recoloring x if necessary. ∎

3. DISCHARGING

We shall complete the proof of the theorem by using discharging in order to obtain a contradiction. Let (V, E, F) be a plane embedding of G. We assign a "charge" $M(\phi)$ to each element $\phi \in V \cup F$, where

$$M(\phi) := \begin{cases} d(\phi) - 6 & \text{if } \phi \in V, \\ 2r(\phi) - 6 & \text{if } \phi \in F, \end{cases}$$

where $r(f)$ denotes the number of edges around face f. Euler's formula $|V| - |E| + |F| = 2$ can be rewritten in the form $(2|E| - 6|V|) + (4|E| - 6|F|) = -12$, which implies that

$$\sum_{\phi \in V \cup F} M(\phi) = \sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2r(f) - 6) = -12. \quad (1)$$

We shall now redistribute the charge, without changing its sum, in such a way that the sum is provably nonnegative, and this contradiction will prove the theorem. Before stating the redistribution rules, we need some definitions.

A 4-face $uvxy$ is *special* if there is a 3-face vxy (and hence $d(x) = 2$). We shall suppose that the embedding of G is chosen so as to minimize the number of special 4-faces. This implies that

$$\text{for each special 4-face } uvxy, \text{ the edge } vy \text{ separates two triangles,} \quad (2)$$

since otherwise we could reduce the number of special 4-faces by moving edge vy across the path vxy.

A *lo-face* of G is a 3-face or a special 4-face; a *hi-face* is any other face, that is, a nonspecial 4-face or a ≥ 5-face. A *lo-triangle* of G is a 3-face in the graph obtained by deleting all 2-vertices of G and their incident edges; thus by Lemma 4 it is either a 3-face of G having no vertices of degree 2, or a triangle uvy containing a 3-face vxy and a special 4-face $uvxy$ with $d(x) = 2$.

By Lemma 1 (a) and (b), it is possible to find a matching in G that pairs off all the 2-vertices with some of the Δ-vertices: in each component of the forest in Lemma 1(b), choose a Δ-vertex v, and match each 2-vertex u with the Δ-vertex w adjacent to u that is further from v. (Note that the endvertices of this forest are all Δ-vertices.) We call w the *master* of u and u the (unique) 2-*dependant* of w.

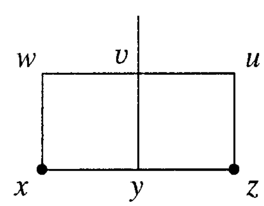

FIGURE 2. Vertex v is a sponsor of vertex y.

If the configuration in Figure 2 occurs in G, where $d(x) = d(z) = 2$, $d(v) \geq 4$ and neither of the 4-faces $vwxy$ and $vuzy$ is special, then we say that v is a *sponsor* of y. Note that y in turn can sponsor other vertices, and even cycles of sponsorship are possible.

The rules for redistribution of charge are as follows.

R1: A special 4-face gives charge 2 to its 2-vertex. A nonspecial 4-face gives 1 to an incident ≤ 5-vertex and 1 to the vertex opposite it. A ≥ 5-face gives 3 to a 2-vertex and 1 to any other ≤ 5-vertex.

R2: Suppose $d(v) = 2$. If v is incident with a ≥ 5-face, then v receives 1 from its master. Otherwise, v receives 13/10 from its master and 7/10 from its other neighbor.

R3: Suppose $d(v) = 3$. If v is not incident with hi-faces, then v receives 1 from each of its neighbors. If v is incident with precisely one hi-face $\cdots avb \cdots$ then v receives 3/5 from each of a, b and 4/5 from its third neighbor. If v is incident with precisely two hi-faces $\cdots avb \cdots$ and $\cdots bvc \cdots$ then v receives 1/2 from each of a, c.

R4: If $d(v) = 4$ or 5 then v receives 1/2 along every incident edge that separates two lo-faces.

R5: A vertex gives 1/5 to any vertex of which it is a sponsor.

R6: If $f = vxwy$ is a nonspecial 4-face with $d(x) = 2$ and $d(y) \geq 10$, so that f gives 1 to y by R1, then y gives 1/2 to each of v and w.

Let $M^*(\phi)$ be the resulting charge on ϕ, so that $\sum_{\phi \in V \cup F} M^*(\phi) = -12$ by (1). We shall obtain a contradiction by proving that $M^*(\phi) \geq 0$ for each element ϕ. We start with the easy cases.

Suppose first that ϕ is a face f. If $r(f) = 3$ then $M^*(f) = M(f) = 0$. If $r(f) = 4$ then $M(f) = 2$ and (by Lemma 1(a)) f gives up at most 2 by R1. If $r(f) = r \geq 5$ then $M(f) = 2r - 6$ and (by Lemma 4) f gives up at most $2 + \lfloor \frac{1}{2} r \rfloor$ by R1. Thus $M^*(f) \geq 0$ in all cases.

Suppose now that ϕ is a vertex v. If $d(v) = 2$ then $M(v) = -4$. If v is incident with a ≥ 5-face f then v receives 3 from f by R1 and 1 from its master by R2. The only other possibilities (since G is simple) are that v is between two (nonspecial) 4-faces, or between a (special) 4-face and a 3-face, in which case v receives 2 from the 4-face(s) by R1 and 2 from neighboring vertices by R2. In either case, $M^*(v) = 0$.

If $d(v) = 3$ then $M(v) = -3$. By R1, v receives 1 from any incident hi-face, and it is easy to see from R3 that $M^*(v) = 0$.

If $d(v) = 4$ or 5 then $M(v) = -2$ or -1. By R1, v receives 1 from any incident hi-face. By this and R4, it is easy to see that $M^*(v) \geq 0$ if v is not a sponsor. If v is a sponsor, then it receives 2 from the two hi-faces (non-special 4-faces) involved in the sponsorship. If there is another hi-face at v then v receives another 1 which suffices for five sponsorships; otherwise, the other faces are all lo-faces and v receives at least $\frac{1}{2}$ between them, which covers its one sponsorship.

If $d(v) = 6$ or 7 then $M(v) \geq 0$ and v gives nothing if it is not a sponsor. If v is a sponsor then it receives 2 from the two hi-faces (non-special 4-faces) involved in the sponsorship, and this suffices for $10 > 7$ sponsorships.

If $d(v) = 8$ or 9 then $M(v) \geq 2$ and v can give $\frac{1}{2}$ to at most four vertices by R4. If v is a sponsor then it receives an additional 2 from two hi-faces, which suffices for $10 > 9$ sponsorships.

If $d(v) \geq 10$ then, in addition to giving charge by R4 and R5, v can give charge to 3-vertices by R3 and (if $d(v) \geq 11$) to 2-vertices by R2. We shall deal with this case in the next section.

4. VERTICES WITH DEGREE AT LEAST 10

Let v be a fixed vertex with $d(v) \geq 10$. For the sole purpose of proving $M^*(v) \geq 0$, we carry out a temporary local redistribution of charge whereby some neighbors of v that have received charge from v will give some of it to other neighbors of v that have received no charge from v. We shall prove that, after this local redistribution,

$$v \text{ gives 1 to its 2-dependant (if any) and at most } \tfrac{2}{5} \text{ to each other neighbor.} \qquad (3)$$

This will suffice to prove the result, since it shows that $M^*(v) \geq (d(v)-6)-1-\frac{2}{5}(d(v)-1) = \frac{3}{5}(d(v)-11) \geq 0$ if $d(v) \geq 11$, while if $d(v) = 10$ then v is not adjacent to any 2-vertices by Lemma 1(a) and so $M^*(v) \geq (d(v)-6) - \frac{2}{5}d(v) = 0$.

Let x be a neighbor of v that has received ''too much'' charge from v (that is, more than is allowed for in (3)). The following rules for the temporary local redistribution of charge cover all possibilities for x. (Note that v cannot be a sponsor of x if $d(x) < \Delta$, by Lemma 1(a) and the structure of Figure 2.)

L1: Suppose that x is a 2-vertex that has received $\frac{13}{10}$ or $\frac{7}{10}$ from v by R2. If edge vx separates a 3-face vxy from a special 4-face $uvxy$, then x now gives $\frac{3}{10}$ to y (and so retains 1 or $\frac{2}{5}$). The only other possibility is that vx separates two nonspecial 4-faces $uvxy$ and $wvxy$, in which case x now gives $\frac{3}{20}$ to each of u, w unless one has degree 3 and the other has larger degree, in which case x gives $\frac{3}{10}$ to whichever has larger degree. (The reason for this is that a 3-vertex could have received charge from v by R3. However, this cannot happen if both u and w have degree 3, since then neither belongs to a triangle by Lemma 2(b).)

L2: Suppose $d(x) = 3$. Suppose first that vx separates two lo-triangles vxy_1 and vxy_2. If xy_1y_2 is a lo-triangle then x has received 1 from v by R3 and now gives $\frac{3}{10}$ to each of y_1, y_2; otherwise x has received $\frac{4}{5}$ from v by R3 and now gives $\frac{1}{5}$ to each of y_1, y_2. Suppose now that vx separates a lo-triangle vxy from a hi-face. If there is another lo-triangle at x then x has received $\frac{3}{5}$ from v by R3 and now gives $\frac{1}{5}$ to y; otherwise x has received $\frac{1}{2}$ from v by R3 and now gives $\frac{1}{10}$ to y. (If vx separates two hi-faces then x has received nothing from v.)

L3: Suppose that $d(x) = 4$ or 5 and x has received $\frac{1}{2}$ from v by R4. Then vx separates two lo-triangles vxy_1 and vxy_2, and x now gives $\frac{1}{20}$ to each of y_1, y_2.

Note that the redistribution in L2 and L3 always happens along an edge xy of a triangle vxy.

It is clear from L1–L3 that (3) holds for any neighbor of v that received charge from v by R1–R6, since no such neighbor receives anything more by L1–L3. So let y be a neighbor of v that receives charge by L1–L3. We must prove that it receives at most $\frac{2}{5}$.

Suppose first that y receives $\frac{3}{10}$ by L1 from a 2-vertex x adjacent to both v and y. Then the path vxy separates a 3-face vxy from a special 4-face $uvxy$, and by (2) the edge vy separates two 3-faces, say vxy and vwy. By Lemma 2(a), $d(u) \geq 4$ and $d(w) \geq 4$. By L3, y receives at most $\frac{1}{20}$ from each of u, w, and nothing from any other vertex except x, making a total of at most $\frac{3}{10} + \frac{2}{20} = \frac{2}{5}$.

Suppose now that y receives $\frac{3}{10}$ or $\frac{3}{20}$ from a 2-vertex x across a 4-face by L1, so that vx lies between nonspecial 4-faces $vxwy$ and $vxwz$, say. If $d(y) = 3$ then y receives $\frac{3}{20}$ from x and at most $\frac{3}{20}$ from a 2-vertex on the other side of vy, making less than $\frac{2}{5}$ in total. If $4 \leq d(y) \leq 9$ then y is not adjacent to 3-vertices by Lemma 1(a); it receives at most $\frac{3}{10}$ from x and nothing more unless it receives $\frac{3}{10}$ from a 2-vertex on the other side of vy, in which case y is a sponsor of v and gives back $\frac{1}{5}$ to v by R5, thus retaining exactly $\frac{2}{5}$ from v. If $d(y) \geq 10$ then y can receive up to $\frac{3}{10}$ from x and $\frac{3}{10}$ from a 2-vertex or 3-vertex on the other side of vy, but y gives $\frac{1}{2}$ to v by R6, and so retains at most $\frac{1}{10}$ from v.

Finally, suppose y receives nothing from 2-vertices. If y is sponsored by v then it receives $\frac{1}{5}$ and nothing more from v. Assume y is not sponsored by v. By L2 and L3, y can get only $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ or $\frac{1}{20}$ from at most two $\leq$ 5-vertices x_1, x_2 lying in triangles vyx_1 and vyx_2. To receive more than $\frac{2}{5}$ in total, y must receive $\frac{3}{10}$ from x_1 or x_2, say x_1, and either $\frac{3}{10}$ or $\frac{1}{5}$ from x_2. By L2, there are three lo-triangles at x_1 and two or three lo-triangles at x_2. But this contradicts Lemma 3, and this contradiction completes the proof of the Theorem.

References

[1] M. Behzad, Graphs and their chromatic numbers, Doctoral Thesis, Michigan State University (1965).

[2] O. V. Borodin, On the total colouring of planar graphs, *J. Reine Angew. Math.* **394** (1989), 180–185.

[3] O. V. Borodin, A. V. Kostochka, and D. R. Woodall, Total colourings of planar graphs with large girth, submitted.

[4] O. V. Borodin, A. V. Kostochka, and D. R. Woodall, List edge and list total colourings of multigraphs, submitted.

[5] P. Erdős, A. L. Rubin, and H. Taylor, Choosability in graphs, in Proc. West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata, 1979, *Congr. Numer.* **26** (1979), 125–157.

[6] T. R. Jensen and B. Toft, *Graph Coloring Problems,* Wiley, New York (1995), 88.

[7] A. V. Kostochka, The total coloring of a multigraph with maximal degree 4, *Discrete Math.* **17** (1977), 161–163.

[8] A. V. Kostochka, Upper bounds of chromatic functions on graphs (in Russian), Doctoral Thesis, Novosibirsk (1978).

[9] A. V. Kostochka, The total chromatic number of any multigraph with maximum degree five is at most seven, *Discrete Math.* **162** (1996), 199–214.

[10] M. Rosenfeld, On the total coloring of certain graphs, *Israel J. Math.* **9** (1971), 396–402.

[11] N. Vijayaditya, On total chromatic number of a graph, *J. London Math. Soc. (2)* **3** (1971), 405–408.

[12] V. G. Vizing, Some unsolved problems in graph theory (in Russian), *Uspekhi Mat. Nauk* **23** (1968), 117–134. English translation in *Russian Math. Surveys* **23,** 125–141.

[13] V. G. Vizing, Vertex colouring with given colours (in Russian), *Metody Diskret. Analiz.* **29** (1976), 3–10.